# GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

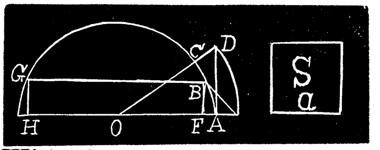
## SOLUTIONS OF PROBLEMS.

37. Proposed by B. F. BURLESON, Oneida Castle, New York.

Inscribe in a semi-circle (1), a rectangle having a given area; (2), a rectangle having the maximum area.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, and S. N. COLLIER, University of Mississippi.

(1). Let $ABC$ be the given semi-circle with center $O$; and let $S$ be a square of given area with side $a$.



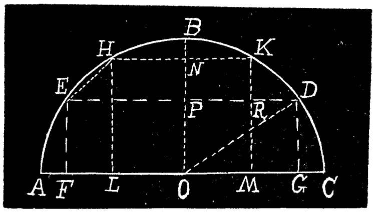

To inscribe in $ABC$ a rectangle equivalent to $S$.

*Construction*:—At $A$, the left-hand extremity of the semi-circumference, draw a tangent $AD$ making it equal to $a$. Draw $OD$. With $O$ as center and $OD$ as radius describe an arc cutting $OA$ produced at $E$. Through $E$ draw a line making an angle of 45° with $EO$, intersecting the circumference at $B$. Through $B$ draw a parallel to $AO$ cutting the circumference at $G$. Through $B$ and $G$ draw perpendiculars to $AO$ meeting the bounding diameter at $F$ and $H$ respectively. Then $FBGH$ is the rectangle required.

*Proof*:—$OD=\sqrt{(AD^2+OR^2)} = \sqrt{(a^2+R^2)}$, denoting $OA$ by $R$. But $OD=OE=OF+FE=OF+FB$. $\therefore$ $\sqrt{(a^2+R^2)}=OF+FB$.

Squaring, $a^2+R^2=OF^2+2.OE.FB+FB^2$. But $OF^2+FB^2=R^2$. $\therefore$ $a^2=2.OF.FB$. $2.OF$ is the base of the rectangle and $FB$ is its altitude. Also, $a^2$=given area. $\therefore$ the rectangle is equivalent to the given area.

(2). Let $ABC$ be a given semi-circle with center $O$. To inscribe in $ABC$ a maximum rectangle.




*Construction*:—Draw the radius $OD$ making an angle of 45° with $OC$. With $D$ as one vertex construct the rectangle $DEFG$. $DEFG$ is the rectangle required.

*Proof:*—Let $LMKH$ be any other inscribed rectangle; $OB$ the radius perpendicular to $OC$.

Compare rectangles $PG$ and $NM$, the halves of the rectangles $EG$ and $HM$. Rectangle $PM$ is common to the two. Rect. $RG>$ rect. $NR$; for $DG(=DP)>RP$, and $DR>KR$.

Since $\angle RKD>\angle RDK$, the former being measured by one-half of an arc greater than 90° and the latter by one-half an arc less than 90°.

It follows that $EDGF$ is the rectangle required.

*A. L. Foote* furnished a neat algebraic solution; *G. B. M. Zerr*, *P. S. Berg*, *Cooper D. Schmitt* solved the problem by the calculus, and *C. D. M. Showalter* gave a good geometrical solution. Space forbids further consideration of this problem.

**38. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.**

Give a *strictly geometric* proof of my fundamental theorem of the Inscription of Regular Polygons, viz: Suppose a circle of unit radius divided at the points $A, A_1, A_2, A_3, \ldots A_p, \ldots$ into $2p+1$ equal parts and the diameter $AO$ drawn. Then, if the chords $OA_1, OA_2, \ldots OA_p$ be drawn, we have $OA_1-OA_2+OA_3-OA_4+OA_5-\ldots\pm OA_p=1$.

**Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

For plainness we will solve this rpoblem in full for the 13-gon. A general solution is as easy, but not as clearly understood.

Let $OA_6, OA_5, OA_4$, &c. $=a_1, a_2, a_3$, &c.

$$A_4A_5=A_5A_6=A_6A_7=\text{\&c}=c,\ A_4A_6=A_5A_7=A_6A_8=\text{\&c.}=d.$$

Now by Ptolemy's Theorem:—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle &c., we easily get the following relations:

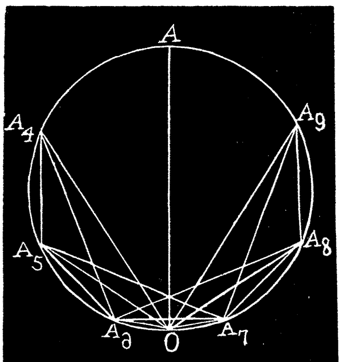
| | | |
|---|---|---|
| $c(a_1+a_3)=da_2$ | $c(a_6+a_8)=da_7$ | $c(a_{10}+a_{12})=da_{11}$ |
| $c(a_2+a_4)=da_3$ | $c(a_7+a_9)=da_8$ | $c(a_{11}+a_{13})=da_{12}$ |
| $c(a_3+a_5)=da_4$ | $c(a_8+a_{10})=da_9$ | $c(a_{12}-a_1)=da_{13}$ |
| $c(a_4+a_6)=da_5$ | $c(a_9+a_{11})=da_{10}$ | $c(a_2-a_{13})=da_1$ |
| $c(a_5+a_7)=da_6$ | | |

Hence $d\{(a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13})-(a_2+a_4+a_6+a_8+a_{10}+a_{12})\}$
$=c\{(a_2-a_{13})+(a_2+a_4)+(a_4+a_6)+(a_6+a_8)+(a_8+a_{10})+(a_{10}+a_{12})+(a_{12}-a_1)$
$-(a_1+a_3)-(a_3+a_5)-(a_5+a_7)-(a_7+a_9)-(a_9+a_{11})-(a_{11}+a_{13})\}$.

$\therefore\ (d+2c)\{(a_2+a_4+a_6+a_8+a_{10}+a_{12})$
$-(a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13})\}=0$.

$\therefore\ a_2+a_4+a_6+a_8+a_{10}+a_{12}$
$=a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13}$.

Generally $a_2+a_4+a_6+\ldots+a_{2p}$
$=a_1+a_3+a_5+\ldots+a_{2p+1}$.




In the above, $O$ can be any point between $OA_p$ and $OA_{p+1}$.

In the problem $OA=a_7=2$, $a_6=a_8$, $a_4=a_6$, $a_2=a_4$, $a_1=a_{13}$, $a_3=a_{11}$, $a_5=a_9$.

$\therefore\ a_2+a_4+a_6=a_1+a_3+a_5+1$, but $a_6=OA_1$, $a_5=OA_2$, $a_4=OA_3$, $a_3=OA_4$, $a_2=OA_5$, $a_1=OA_6$.

$\therefore\ OA_1-OA_2+OA_3-OA_4+OA_5$
$-OA_6=1$, $p$, even.

For 15-gon. $OA_1-OA_2+OA_3-OA_4+OA_5-OA_6+OA_7=1$, $p$, odd.

$\therefore\ OA_1-OA_2+OA_3-OA_4+OA_5-\ldots\pm OA_p=1$, as $p$ is odd or even.

**39. Proposed by J. K. ELLWOOD, Principal of Colfax Schools, Pittsburg, Pennsylvania.**

If on the three sides of any plane triangle equilateral triangles be described, the lines joining the centres of these equilateral triangles form an equilateral triangle.